PHYS4150 — PLASMA PHYSICS

LECTURE 2 - PLASMA PROPERTIES: DENSITY AND TEMPERATURE

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Plasma properties: Density and Temperature.

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1 PLASMA PROPERTIES

COMPOSITION: ions and electrons

NUMBER DENSITY: ions and electrons in laboratory plasmas $\sim 10^8 - 10^{14}\mathrm{cm}^{-3}$

TEMPERATURE: measured in electron Volts (eV), $1e\mathrm{V} = 11,600\mathrm{K}$

$e \cdot U = \frac{m}{2}v^2 = \mathrm{k_B}T = e \cdot 1\mathrm{V}$
$1e\mathrm{V} = 1.602 \cdot 10^{-19}\mathrm{C} \cdot \mathrm{J/C}$
$1e\mathrm{V} = 1.602 \cdot 10^{-19}\mathrm{J}$

DISTANCE SCALE: Debye length λ_D

TIME SCALE: plasma frequency $\omega_p = 2\pi f_p$

VELOCITY SCALE: thermal velocity $v_{th} = \sqrt{\frac{8kT}{\pi m}}$

2 REVIEW: THERMODYNAMICS

Let us starting with a review of some important thermodynamical principles.

2.1 *First law of thermodynamics*

The *First Law of Thermodynamics* states that the change of the internal energy U is given by the sum of the work δW and heat δQ exchanged with the environment:

$$dU = \delta W + \delta Q \quad (1)$$

U is an extensive state function and a thermodynamical potential

Note the use of δ instead of d. This indicates that the amount of exchanged heat

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and work does depend on how the thermodynamical process is performed, and thus, δW and δQ are not *exact differentials*. In contrast, the change of the interior energy depends only on the initial and final state and is therefore an *exact differential*.

2.2 *Second law of thermodynamics*

The *Second Law of Thermodynamics* is closely related to the *entropy*, which is defined as the reversibly exchanged heat at constant temperature T

S is an extensive state function, while T is an intensive state function

$$dS = \frac{\delta Q}{T}. \tag{2}$$

The second law says now that for a closed system at equilibrium the entropy does not change, i.e.

$$dS = 0. \tag{3}$$

At a given temperature the amount of irreversibly exchanged heat is always smaller than the amount of reversibly exchanged heat, and thus

$$\delta Q_{irr} < \delta Q_{rev} = T\,dS. \tag{4}$$

For a closed system at equilibrium the entropy takes its maximum value S_{max}, while for an irreversible process $dS > 0$.

2.3 *Ideal gas*

In an ideal gas the particles are assumed to undergo only elastic collisions. In this case the equation of state is

$$pV = N\mathrm{k_B}T, \tag{5}$$

where p, V, and N are the pressure, volume, and particle number of the gas. The Boltzmann constant $\mathrm{k_B}$

$$\mathrm{k_B} = 1.308 \cdot 10{-}23\mathrm{J/K} = 8.617 \cdot 10^{-5} e\mathrm{V}. \tag{6}$$

relates the average kinetic energy of the gas with the temperature. For an ideal gas the average (translational) energy is

$$\frac{1}{2}m\langle\, v^2 \,\rangle = \frac{3}{2}\mathrm{k_B}T. \tag{7}$$

3 DENSITY

SOLID As an example let us consider aluminum which has a density of $\rho_{Al} = 3 \cdot 10^3 \mathrm{kg/m}^3$ and an atomic mass of $m_{Al} = 27\mathrm{u}$. We now want to find the number of aluminum atoms per unit volume:

$u = 1.66 \cdot 10^{-27}\mathrm{kg}$ is the atomic mass unit.

$$n_{Al} = \frac{\rho_{Al}}{m_{Al} u} = \frac{3 \cdot 10^3 \mathrm{kg/m}^3}{27 \cdot 1.66 \cdot 10^{-27} \mathrm{kg}} = 6.8 \cdot 10^{28} \mathrm{m}^{-3}. \tag{8}$$

AIR At standard pressure one mol of air has a volume of $22.4\mathrm{l} = 22.4 \cdot 10^{-3} \mathrm{m}^3$. One mol are $6 \cdot 10^{23}$ particles, and thus

$$n_{air} = \frac{6 \cdot 10^{23}}{22.4 \cdot 10^{-3} \mathrm{m}^3} = 2.7 \cdot 10^{25} \mathrm{m}^{-3}. \tag{9}$$

	$n[\mathrm{m}^{-3}]$	kT [eV]
Solar wind @ Earth	5	50
ionosphere	$10^5 - 10^6$	0.02
Solar corona	10^6	100
tokamak	10^{14}	10^4
laser-produced	10^{20}	100
glow discharge	$10^8 - 10^{10}$	2

4 TEMPERATURE

Let us have a closer look at the velocity distribution $f(\mathbf{v})$ of a gas and how it relates to its temperature. Because the gas motion is isotropic, $f(\mathbf{v})$ can only be a function of $\mathbf{v}^2$. On the other hand, the components of $f(\mathbf{v})$ must be independent, which implies that

$$f(\mathbf{v}^2) = f(v_x^2 + v_y^2 + v_z^2) = f(v_x^2) f(v_y^2) f(v_z^2). \tag{10}$$

The only function that fulfills Eq. (10) is

$$f(\mathbf{v}^2) = c \cdot e^{a\mathbf{v}^2}. \tag{11}$$

To find the constant c we require that the components of f are normalized, i.e. $\int f_i(v_i)\, dv = 1$, which is only possible if $a < 0$, and

$$1 = c \int e^{-av^2}\, dv = c\sqrt{\frac{\pi}{a}}. \tag{12}$$

To obtain the constant a we note that in a gas at equilibrium the energy per degree of freedom is $\frac{1}{2}\mathrm{k_B}T$, and therefore

$$\mathrm{k_B}T = m\langle v_i^2 \rangle = m\int v_i^2 f(v_i)d\,v_i = m\sqrt{\frac{\pi}{a}}\int \exp\left\{-av_i^2\right\} v_i^2 d\,v_i. \tag{13}$$

$dv_i = \frac{1}{2\sqrt{a}}\frac{dx}{\sqrt{x}}$

Replacing the argument of the exponential by $x = av_i^2$ we get

$$\mathrm{k_B}T = \frac{m}{\sqrt{\pi}a}\int_0^\infty e^{-x}\sqrt{x}dx = \frac{m}{\sqrt{\pi}a}\Gamma(\frac{3}{2}), \tag{14}$$

where the *Gamma function* $\Gamma(x)$ is defined as

$$\Gamma(z) = \int_0^\infty e^{-x}x^{z-1}dx \tag{15}$$

$$\Gamma(z+1) = \Gamma(z)\cdot z \tag{16}$$

$$\Gamma(1) = 1 \tag{17}$$

$$\Gamma(\frac{1}{2}) = \sqrt{\pi} \tag{18}$$

From this follows that $\Gamma(3/2) = \frac{\sqrt{\pi}}{2}$ and

$$f(v) = \sqrt{\frac{m}{2\pi\mathrm{k_B}T}}\exp\left\{-\frac{mv^2}{2\mathrm{k_B}T}\right\}$$

$$f(\mathbf{v}) = \left\{\frac{m}{2\pi\mathrm{k_B}T}\right\}^{3/2}\exp\left\{-\frac{m\mathbf{v}^2}{2\mathrm{k_B}T}\right\}$$